Solution of Initial Value Problem for Ordinary Differential Equations by Circular Splines

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Abstract

The paper presents a method of numerical solving the initial value problem for ordinary differential equations based on some non-linear difference formula of implicit type. In each of successively considered subintervals of a given interval the approximate solution is computed iteratively according to the formula obtained by replacing the exact solution with a circular arc. The approximate solution is a C^1 function composed of arcs segments, commonly known as circular arc spline, circular spline or arc spline. The discussed method is a one-step method of second order.

Key words: arc spline, circular spline, differential equation.

Introduction

In the simplest case the initial value problem for ordinary differential equations is to find a real function $y = y(x)$ defined and differentiable on the interval $[a, b]$, the derivative $y' = y'(x)$ of which fulfills the equation

$$y' = f(x, y) \tag{1}$$

and the function itself satisfies the initial condition

$$y(a) = y_0. \tag{2}$$

To guarantee the existence and uniqueness of a solution of the initial value problem (1–2) some assumptions about the regularity of the function f should be accepted [3]. Usually it is assumed that f is continuous with respect to both variables x and y on some closed domain G containing $[a, b]$, and that it satisfies on G the Lipschitz condition

$$|f(x, y) - f(x, y^*)| \leq L|y - y^*|, \quad x \in [a, b] \tag{3}$$

for a certain positive constant L.

Numerical methods for solving the initial value problem can be divided into giving an approximate analytical solution in the whole range of $[a, b]$, and finding

*Rev. March 21, 2018, e-mail: `kazimierz@jakubczyk.net` numerical approximations y_i to the values $y(x_i)$ of the exact solution in a finite number of points

$$x_i = a + ih \quad (i = 0, 1, \ldots, n), \tag{4}$$

where $h = (b - a)/n$ for a positive integer constant n. These points are called ***knots*** or ***nodes***. Of course, it is possible to omit the assumption that they are equally spaced in $[a, b]$ and employ a variable step-size.

Examples of methods of the first group are the methods associated with Taylor series [3] or Chebyshev polynomials and series [4], and the second — one-step Euler and Runge-Kutty or multi-step Adams methods [3, 5]. Although the methods of the second group give only an array of approximations y_i to the values $y(x_i)$ of exact solution in knots they allow, however, to build an analytical approximation in the entire range, e.g. by interpolating of data (x_i, y_i) by polynomial spline [3]. Some of these methods, based on linear difference formulas derived from the polynomial interpolation, give the analytical approximation of a such type directly. For example, a second order predictor-corrector method, described using the explicit Euler formula (predictor)

$$y_{i+1}^{(0)} = y_i + hf(x_i, y_i), \tag{5}$$

and the implicit Adams-Moulton formula (corrector)

$$y_{i+1}^{(k+1)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(k)})\right] \quad (i = 0, 1, \ldots, n-1), \tag{6}$$

determines the iterative procedure of computing the approximate value of y_{i+1}, when the value y_i has been determined previously. The corrective formula (6), used for successive $k = 0, 1, 2, \ldots$ until the required accuracy is achieved, results from the linear differentiation formula

$$y_{i+1} = y_i + \frac{h}{2}\left(y_i' + y_{i+1}'\right) \quad (i = 0, 1, \ldots, n-1), \tag{7}$$

that is known as trapezoidal rule for integration and is exact for second degree polynomials. Thus, the method gives a solution in the form of a spline function, which in each of the subintervals $[x_i, x_{i+1}]$ is a polynomial of a degree not higher than 2, and in the entire $[a, b]$ is continuous with its first order derivative. The method's order is equal to 2.

Non-linear difference formula

It is well known that circles, like parabolas, have three degrees of freedom (we assume that the straight line is a circle of radius $r = \infty$). This suggests an idea to replace the trapezoidal rule (7) by a differentiation formula fulfilled by circular arcs, and in turn leads to the method giving the analytical approximate solution of (1–2) in the form of a circular spline.

Definition 1. *The function $s \in C^1[a, b]$ is said to be a* ***circular spline*** *with knots (4) if in each of the subintervals $[x_i, x_{i+1}]$ specified by these knots it is a circular arc lying either on the upper or on the lower half of a circle.*

Thus, s is a piecewise circular arc consisting of n arcs. In each interior knot x_i two adjacent arcs, the one on the left and the one on the right, are joined tangentially in x_i, but they may not have a common second order derivative there. Within each subinterval $[x_i, x_{i+1}]$, the corresponding arc is continuous with its all derivatives, so s may not have only higher order derivatives than the first in interior knots.

It is worth mentioning that in a slightly more general definition of a circular spline, the first order derivative of this function can take infinite $-\infty$ and $+\infty$ values in knots [2].

Returning to the differentiation formula fulfilled by circular arcs, analogous to the trapezoidal rule for parabolas, it turns out that such a formula exists.

Theorem 1. *Let y_i and y'_i be any real numbers $(i = 0, 1, \ldots, n)$. There exists a unique circular spline s on $[a, b]$ with knots (4) that fulfills the interpolation conditions*

$$y_i = s(x_i), \quad y'_i = s'(x_i) \quad (i = 0, 1, \ldots, n), \tag{8}$$

if and only if

$$y_{i+1} = y_i + hB(y'_i, y'_{i+1}) \quad (i = 0, 1, \ldots, n-1), \tag{9}$$

where for any real numbers u, v

$$B(u, v) = \frac{v\sqrt{1+u^2} + u\sqrt{1+v^2}}{\sqrt{1+u^2} + \sqrt{1+v^2}}. \tag{10}$$

Proof. Without loss of generality we can limit the considerations, to focuss our attention, to the subinterval $[x_1, x_2]$. It is not difficult to notice that in the case of $y'_1 = y'_2$ the relation (9) simplifies to

$$y_2 = y_1 + h \cdot y'_1 \quad \text{or} \quad y_2 = y_1 + h \cdot y'_2$$

which is equivalent to the existence of a unique stright line segment (degenerate circular arc of radius $r = \infty$) satisfying interpolation conditions (8) for $i = 1, 2$. Thus, it remains to prove the theorem for $y'_1 \neq y'_2$.

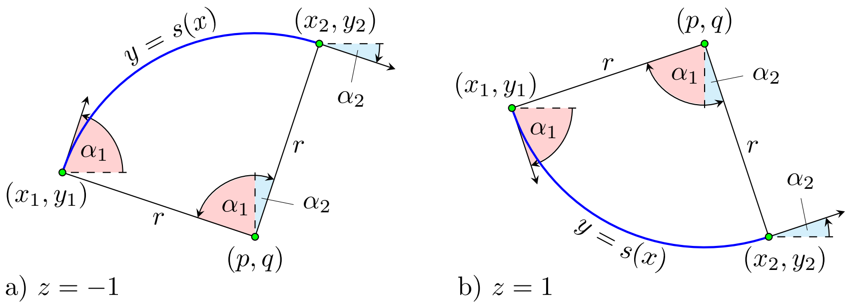

Figure 1: Circular arc $y = s(x)$ for $x \in [x_1, x_2]$

Necessity. As can be seen in Figure 1, the circular arc $y = s(x)$, defined on $[x_1, x_2]$, is a part of a circle of radius r and the centre (p, q). It is obvious that end-points (x_1, y_1) and (x_2, y_2) of the arc are both lying either on the upper or on the lower half of the circle. Assume that $z = -1$ if they are on the upper half (Fig. 1a), and $z = 1$ if on the lower half (Fig. 1b). Moreover, let α_1 be the angle beetwen the positive x-axis and the tangent to the arc at (x_1, y_1), and let α_2 be the angle beetwen the positive x-axis and the tangent to the arc at (x_2, y_2). Both of these tangents are given by

$$y - y_1 = y_1'(x - x_1), \quad y - y_2 = y_2'(x - x_2). \tag{11}$$

Because angles can be measured in different ways we can assume that α_1 and α_2 are in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$ and that the signs of them are the same as the signs of y_1' and y_2', respectively. It is easy to verify that

$$p = x_i - zr\sin\alpha_i, \quad q = y_i + zr\cos\alpha_i \quad (i = 1, 2).$$

Since sines and cosines in the above equalities can be expressed in terms of y_i', it follows that

$$p = x_i - zr\frac{y_i'}{\sqrt{1 + {y_i'}^2}}, \quad q = y_i + zr\frac{1}{\sqrt{1 + {y_i'}^2}} \quad (i = 1, 2).$$

Hence

$$x_2 - x_1 = zr\left(\frac{y_2'}{\sqrt{1 + {y_2'}^2}} - \frac{y_1'}{\sqrt{1 + {y_1'}^2}}\right) \tag{12}$$

and

$$y_2 - y_1 = zr\left(\frac{1}{\sqrt{1 + {y_1'}^2}} - \frac{1}{\sqrt{1 + {y_2'}^2}}\right). \tag{13}$$

Eliminating zr by simple transformation we obtain

$$y_2 - y_1 = (x_2 - x_1)B(y_1', y_2'), \tag{14}$$

where B is a function of two real variables specified by (10).

Sufficiency. Assume that equality (14) is satisfied. Let also, as specified in the previous part of the proof, α_1 and α_2 be the angles of inclination of straight lines (11). Moreover, let β be the angle beetwen the positive x-axis and the stright line passing through the points (x_1, y_1) and (x_2, y_2). We can assume, as in the case of α_1 and α_2, that $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and that the sign of β is the same as the sign of the difference quotient $(y_2 - y_1)/(x_2 - x_1)$, that is, in relation to $x_1 < x_2$, as the sign of $y_2 - y_1$. By manipulating (14) and (10) we obtain

$$\frac{y_2 - y_1}{x_2 - x_1} = B(y_1', y_2') = \frac{y_2'\sqrt{1 + {y_1'}^2} + y_1'\sqrt{1 + {y_2'}^2}}{\sqrt{1 + {y_1'}^2} + \sqrt{1 + {y_2'}^2}} =$$

$$= \frac{\dfrac{y_2'\sqrt{1 + {y_1'}^2} + y_1'\sqrt{1 + {y_2'}^2}}{\sqrt{1 + {y_1'}^2}\sqrt{1 + {y_2'}^2}}}{\dfrac{\sqrt{1 + {y_1'}^2} + \sqrt{1 + {y_2'}^2}}{\sqrt{1 + {y_1'}^2}\sqrt{1 + {y_2'}^2}}} = \frac{\dfrac{y_1'}{\sqrt{1 + {y_1'}^2}} + \dfrac{y_2'}{\sqrt{1 + {y_2'}^2}}}{\dfrac{1}{\sqrt{1 + {y_1'}^2}} + \dfrac{1}{\sqrt{1 + {y_2'}^2}}}.$$

Next, using trigonometric functions, we get

$$\tan\beta = \frac{\sin\alpha_1 + \sin\alpha_2}{\cos\alpha_1 + \cos\alpha_2} = \frac{2\sin\dfrac{\alpha_1+\alpha_2}{2}\cos\dfrac{\alpha_1-\alpha_2}{2}}{2\cos\dfrac{\alpha_1+\alpha_2}{2}\cos\dfrac{\alpha_1-\alpha_2}{2}} = \tan\frac{\alpha_1+\alpha_2}{2}.$$

All the values of α_1, α_2 and β in the above equation are between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so it can be seen as relationship of angles

$$\alpha_1 - \beta = \beta - \alpha_2. \tag{15}$$

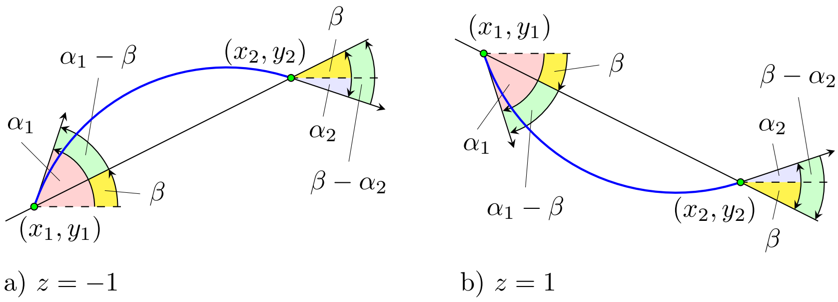

Figure 2: Relationship of angles at end-points of circular arc

The equality (15) is shown graphically in Figure 2. The consequence of it is the existence of a circle that passes through the points (x_1, y_1) and (x_2, y_2), and is tangent at these points to two stright lines (11). The points (x_1, y_1) and (x_2, y_2) intersect the circle into two arcs. By virtue of the range of α_1, α_2 and β the minor arc is a function on $[x_1, x_2]$ that satisfies the interpolation conditions (8) for $i = 1, 2$. If $\alpha_1 > \beta$ (and also $\alpha_2 < \beta$) it lies on the upper half of the circle (Fig. 2a), and if $\alpha_1 < \beta$ (and $\alpha_2 > \beta$) on the lower half (Fig. 2b). The radius r of the arc and the parameter z, indicating the appriopriate half of the circle, can be determined from (12) or (13).

Uniqueness. From the properties of circles it follows that, for $x_1 < x_2$, any three values y_1, y_2, y_1' or y_1, y_2, y_2' are enough to determine the only one circle that passes through the points (x_1, y_1) and (x_2, y_2), and is tangent at (x_1, y_1) to the first stright line of (11) or at (x_2, y_2) to the second one. □

In the sequel we shall need some estimates of the function $y = B(u, v)$ and its first and second order partial derivatives with respect to v (a slightly different form of this function and its estimation is given in [1]).

Lemma 1. *For any real numbers u, v the following inequalities hold:*

$$u - \sqrt{1+u^2} < B(u, v) < u + \sqrt{1+u^2}, \tag{16}$$

$$0 < \frac{\partial B(u, v)}{\partial v} < 2, \tag{17}$$

$$\left|\frac{\partial^2 B(u, v)}{\partial v^2}\right| < \frac{3}{2}. \tag{18}$$

Proof. Differentiating B with respect to v by simple manipulating we obtain

$$\frac{\partial B(u,v)}{\partial v} = \frac{\sqrt{1+u^2}}{\sqrt{1+v^2}} \frac{\sqrt{1+u^2}\sqrt{1+v^2}+uv+1}{(\sqrt{1+u^2}+\sqrt{1+v^2})^2}. \tag{19}$$

Hence, due to inequality

$$|uv|+1 \leq \sqrt{1+u^2}\sqrt{1+v^2}, \tag{20}$$

we can write

$$0 < \frac{\sqrt{1+u^2}}{\sqrt{1+v^2}} \frac{|uv|+1+uv+1}{(\sqrt{1+u^2}+\sqrt{1+v^2})^2} \leq \frac{\partial B(u,v)}{\partial v} \leq$$

$$\leq \frac{\sqrt{1+u^2}}{\sqrt{1+v^2}} \frac{2\sqrt{1+u^2}\sqrt{1+v^2}}{(\sqrt{1+u^2}+\sqrt{1+v^2})^2} = \frac{2(1+u^2)}{(\sqrt{1+u^2}+\sqrt{1+v^2})^2} < 2.$$

This gives the desired estimate (17).

The left-hand side of (17) implies that B is a stricty increasing function of v for any fixed u. By virtue of

$$\lim_{v\to\pm\infty} B(u,v) = u \pm \sqrt{1+u^2},$$

it means that the estimate (16) is fulfilled.

Differentiating (19) with respect to v, as a result of strenuous algebraic transformations we have

$$\frac{\partial^2 B(u,v)}{\partial v^2} = \frac{\sqrt{1+u^2}(u-v)}{(\sqrt{1+v^2})^3(\sqrt{1+u^2}+\sqrt{1+v^2})^2} - \frac{2\sqrt{1+u^2}v(\sqrt{1+u^2}\sqrt{1+v^2}+uv+1)}{(1+v^2)(\sqrt{1+u^2}+\sqrt{1+v^2})^3}. \tag{21}$$

Hence, by (20) and

$$4\sqrt{1+u^2}\sqrt{1+v^2} \leq (\sqrt{1+u^2}+\sqrt{1+v^2})^2,$$

and due to other simple inequalities, such as

$$|u| < \sqrt{1+u^2}, \quad |v| < \sqrt{1+v^2}, \quad \frac{|v|}{1+v^2} \leq \frac{1}{2},$$

we finally obtain

$$\left|\frac{\partial^2 B(u,v)}{\partial v^2}\right| \leq \frac{\sqrt{1+u^2}(|u|+|v|)}{(\sqrt{1+u^2}+\sqrt{1+v^2})^2} + \frac{4\sqrt{1+u^2}|v|\sqrt{1+u^2}\sqrt{1+v^2}}{(1+v^2)(\sqrt{1+u^2}+\sqrt{1+v^2})^3} <$$

$$< \frac{\sqrt{1+u^2}(\sqrt{1+u^2}+\sqrt{1+v^2})}{(\sqrt{1+u^2}+\sqrt{1+v^2})^2} + \frac{\sqrt{1+u^2}(\sqrt{1+u^2}+\sqrt{1+v^2})^2}{2(\sqrt{1+u^2}+\sqrt{1+v^2})^3} <$$

$$< \frac{\sqrt{1+u^2}}{\sqrt{1+u^2}+\sqrt{1+v^2}} + \frac{\sqrt{1+u^2}}{2(\sqrt{1+u^2}+\sqrt{1+v^2})} < \frac{3}{2},$$

which gives (18) and completes the proof. □

Circular spline solution

We shall construct a continuous and differentiable function s that consists of arcs on subintervals determined by knots (4), and is the approximate solution of the initial value problem (1–2). We require that $s(x_0) = y(x_0)$ and $s'(x_0) = y'(x_0)$, where y is the exact solution of (1–2). So at the beginning we have at x_0 two known values $y_0 = s(x_0)$ and $y'_0 = s'(x_0)$. Starting with them we determine successively, for each $i = 0, 1, \ldots, n-1$, values

$$y_{i+1} = s(x_{i+1}), \quad y'_{i+1} = s'(x_{i+1})$$

by requiring that they satisfy equation (1), and that the values y_i, y_{i+1}, y'_i and y'_{i+1} meet the formula (9) which guarantees the existence of a suitable circular arc on $[x_i, x_{i+1}]$. In this way we find a circular spline such that

$$y'_i = f(x_i, y_i) \quad (i = 0, 1, \ldots, n). \tag{22}$$

The following theorem states that such a construction of a circular spline approximating the exact solution is realizable.

Theorem 2. *If knots* (4) *are equally spaced with the step-size* $h < \frac{1}{2L}$*, where* L *is a constant in the Lipschitz condition* (3)*, then there exists a unique circular spline* s *with knots* (4) *that satisfies indentities* (22)*.*

Proof. Let $i \in \{0, 1, \ldots, n-1\}$, and let y_i and y'_i be the values of the circular spline s and its derivative s' at x_i satisfying the identity $y'_i = f(x_i, y_i)$ (initially, by the assumptions of the construction of s, this equality is met in x_0). We want to determine a circular arc being the next segment of the function s on $[x_i, x_{i+1}]$. It means that y_i, y'_i, y_{i+1} and y'_{i+1} should satisfy the relation (9), and at the same time y_{i+1} and y'_{i+1} should satisfy

$$y'_{i+1} = f(x_{i+1}, y_{i+1}). \tag{23}$$

Both of these requirements can be met if and only if

$$y_{i+1} = y_i + hB(y'_i, f(x_{i+1}, y_{i+1})). \tag{24}$$

Denoting the right-hand member of (24) by $g_h(y_{i+1})$ and applying mean-value theorem, in view of (3) and (17), for any u, v and a certain θ_i we have

$$\begin{aligned} |g_h(u) - g_h(v)| &= h|B(y'_i, f(x_{i+1}, u)) - B(y'_i, f(x_{i+1}, v))| = \\ &= h\frac{\partial B(y'_i, \theta_i)}{\partial v}|f(x_{i+1}, u) - f(x_{i+1}, v)| \leq 2Lh|u - v|. \end{aligned}$$

Thus, if $h < \frac{1}{2L}$, then g_h is a strong contraction mapping. Applying Banach's fixed-point theorem we deduce that the equation (24) has a unique solution y_{i+1}, which may be found by iteration. Next, having given the values y_i, y'_i and y_{i+1}, we can compute y'_{i+1} using (23). So the theorem is established. □

Based on the formula (24), we can determine the implicit method of finding the analytical approximate solution of (1–2) in the form of a circular spline consisting of n arcs. At the begining of each step $i \in \{0, 1, \ldots, n-1\}$, the approximation y_i to the value $y(x_i)$ of the exact solution has already been found (y_0 is given by the initial condition (2)), and also y'_i towards (22). So we can compute the initial prediction $y_{i+1}^{(0)}$ to $y(x_{i+1})$ by using the explicit Euler formula (5). Then, using the correction formula

$$y_{i+1}^{(k+1)} = y_i + hB(y'_i, f(x_{i+1}, y_{i+1}^{(k)})) \tag{25}$$

we continue corrective process for $k = 0, 1, \dots$ until, within the limits of the given accuracy, the results of two successive computations of the values y_{i+1} coincide. Now, having given the quantities y_i, y'_i, y_{i+1} and y'_{i+1} computed in accordance with (23), we can determine the parameters r and z of the sought circular arc using the appropriate equation resulting from (12) or (13).

The initial prediction $y_{i+1}^{(0)}$ can be found by any numerical method of low order. The use of more precise methods is not recommended because, as shown below, the considered method is of the second order.

One of the numerical issues is to determine the moment of stopping the iterations curried out in accordance with the formula (25). It seems sensible to demand that the difference between two succesive approximations of y_{i+1} are acceptable small, that is

$$\left| y_{i+1}^{(k+1)} - y_{i+1}^{(k)} \right| \le \epsilon,$$

where ϵ is some preassigned tolerance. It is worth noting that the number of needed iterations is given by Banach's theorem itself. As the function g_h in the proof of Theorem 2 is a strong contraction mapping with a constant $2Lh < 1$, the approximation error in the $(k+1)$-th iteration is

$$\left| y_{i+1}^{(k+1)} - y_{i+1} \right| \le \frac{2Lh}{1-2Lh} \left| y_{i+1}^{(k+1)} - y_{i+1}^{(k)} \right|.$$

This, however, requires more accurate knowledge of the L factor, which can be estimated during iteration process (see [5], p. 216). Indeed, from the Lipschitz condition (3) it follows that in each iterative step

$$L \approx \left| \frac{f(x_{i+1}, y_{i+1}^{(k+1)}) - f(x_{i+1}, y_{i+1}^{(k)})}{y_{i+1}^{(k+1)} - y_{i+1}^{(k)}} \right| = \left| \frac{{y'}_{i+1}^{(k+1)} - {y'}_{i+1}^{(k)}}{y_{i+1}^{(k+1)} - y_{i+1}^{(k)}} \right|.$$

Example 1. The initial value problem

$$y' = 2xe^{-y}, \quad y(0) = 0 \tag{26}$$

has the unique solution

$$y = \ln(x^2 + 1), \quad x \in (-\infty, \infty).$$

For comparison, we have found, using (5) and (25), the approximate solution of (26) in the form of the circular spline s with 9 knots equally spaced in the interval [0,4], that is for $n = 8$ and $h = \frac{1}{2}$. Both these analytical solutions, $y = y(x)$ and $y = s(x)$ for $x \in [0, 4]$, are shown graphically in Figure 3. Although the step-size seems too large the iterative process is quite quickly convergent (from a few to a dozen or so iterations with accuracy $\epsilon = \frac{1}{2} \times 10^{-8}$), and visually, the approximate solution seems to fit reasonably well, especially in the second half of the range under consideration.

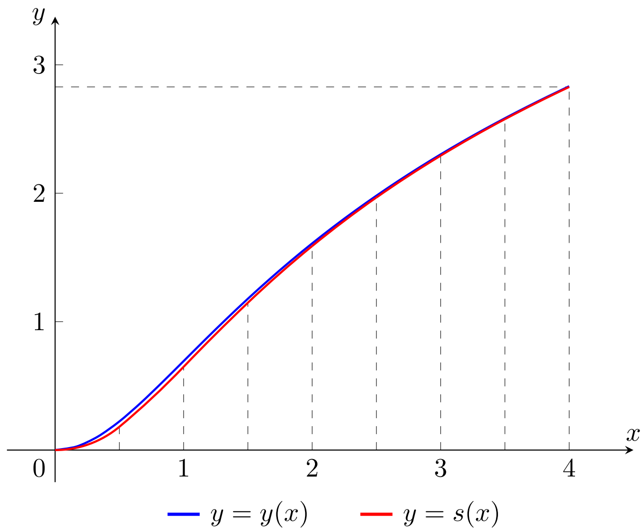

Figure 3: The exact and approximate solutions of (26)

Numerical parameters of all arcs that make up the approximate circular spline with step-size $h = \frac{1}{2}$ on $[0, 4]$ are given in Table 1. Due to the known exact solution, it also provides data for estimating the error of the approximate solution. As can be seen, the maximum approximation error is 0.04474. Similar computations carried out for twice smaller step-size show a clear decrease in error, and so, it is 0.01163 for $h = \frac{1}{4}$, 0.00288 for $h = \frac{1}{8}$ and 0.00072 for $h = \frac{1}{16}$. It seems that a two-fold decrease in step-size leads to a four-fold reduction in error, which allows us to suppose that the accuracy of the method is $O(h^2)$. Also the number of iterations is reduced while the step-size decreases.

i	x_i	y_i	y'_i	$y(x_i)$	$y(x_i) - y_i$	r_i	z_i
0	0.0	0.00000	0.00000	0.00000	0.00000	1.56100	1
1	0.5	0.18118	0.83428	0.22314	0.04196	12.17646	1
2	1.0	0.64841	1.04575	0.69315	0.04474	30.22604	-1
3	1.5	1.14740	0.95238	1.17865	0.03125	17.53497	-1
4	2.0	1.58857	0.81687	1.60944	0.02086	16.82274	-1
5	2.5	1.96684	0.69949	1.98100	0.01416	18.20011	-1
6	3.0	2.29270	0.60596	2.30259	0.00988	20.60703	-1
7	3.5	2.57691	0.53206	2.58400	0.00709	23.75317	-1
8	4.0	2.82801	0.47304	2.83321	0.00520		

Table 1: Approximate solution of (26) and its accuracy

Here $f(x, y) = 2xe^{-y}$, so by mean-value theorem

$$|f(x, y) - f(x, y^*)| = |2x(e^{-y} - e^{-y^*})| = 2xe^{-\xi}|y - y^*| \leq 8|y - y^*|$$

for any $x \in [0, 4]$, $y, y^* \geq 0$ and a certain ξ lying between y and y^*. Hence $L = 8$. Thus, the step-size h is inadequate in view of Theorem 2, as the condition $h < \frac{1}{2L} = \frac{1}{16}$ is expected to be met. But the maximum estimated value of L is about 1, so the step-size $h = \frac{1}{2}$ seems acceptable.

All numerical methods for solving the initial value problem (1–2) can be used without significant changes to the system of ordinary differential equations of the first order, as well to the differential equations of higher-order and their systems. The transition to a multidimensional case relies mainly on replacing the scalar quantities with vectors, and in estimates on replacing the number modules with vector norms.

Example 2. One of the two solutions of the second order differential equation

$$x^2y'' + xy' + x^2y = 0$$

is the Bessel J_0 function specified on all complex plane [4]. This function satisfies the equalities

$$J_0(0) = 1, \quad J_0'(0) = 0, \quad J_0''(0) = -\frac{1}{2}.$$

Therefore in the field of real numbers it can be treated as the solution of ordinary differential equation

$$y'' = -\frac{y'}{x} - y \tag{27}$$

with two initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{28}$$

Note that equation (27) has a singularity at $x = 0$. However, it is removable, as in accordance with the l'Hospital's rule it follows that

$$\lim_{x\to 0} \frac{y'(x)}{x} = \lim_{x\to 0} \frac{J_0'(x)}{x} = \lim_{x\to 0} \frac{J_0''(x)}{1} = -\frac{1}{2}.$$

The initial value problem (27–28) is of the second order, so to solve it by any numerical method for single first order equations, it should be converted into an equivalent system of two first order equations. In general, starting with a few auxiliary notations of functions and values

$$y_1(x) = y(x), \quad y_2(x) = y'(x), \quad y_{1,0} = y_1(a), \quad y_{2,0} = y_2(a)$$

and introducing vector symbols

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix}, \quad \mathbf{f}(x, \mathbf{y}) = \begin{bmatrix} f_1(x, y_1, y_2) \\ f_2(x, y_1, y_2) \end{bmatrix}$$

we can write a second order differential equation together with its two initial conditions in the compact form

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \quad \mathbf{y}'(a) = \mathbf{y}_0 \tag{29}$$

which is very close to (1–2). Just like in one-dimensional case, having given knots (4) equally spaced in $[a, b]$, we want to find a differentiable function $\mathbf{s}(x)$ for $x \in [a, b]$, such that the values

$$\mathbf{y}_i = \begin{bmatrix} y_{1,i} \\ y_{2,i} \end{bmatrix} = \begin{bmatrix} s_1(x_i) \\ s_2(x_i) \end{bmatrix} = \mathbf{s}(x_i), \quad \mathbf{y}_i' = \begin{bmatrix} y_{1,i}' \\ y_{2,i}' \end{bmatrix} = \begin{bmatrix} s_1'(x_i) \\ s_2'(x_i) \end{bmatrix} = \mathbf{s}'(x_i)$$

satisfy

$$\mathbf{y}_i' = \mathbf{f}(x_i, \mathbf{y}_i) \quad (i = 0, 1, \ldots, n) \tag{30}$$

and

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\,\mathbf{B}(\mathbf{y}_i', \mathbf{y}_{i+1}') = \\ = \begin{bmatrix} y_{1,i} \\ y_{2,i} \end{bmatrix} + h \begin{bmatrix} B(y_{1,i}', y_{1,i+1}') \\ B(y_{2,i}', y_{2,i+1}') \end{bmatrix} \quad (i = 0, 1, \ldots, n-1). \tag{31}$$

The equalities (30) mean that the function $\mathbf{s}$ is the approximate solution to the problem (29) in knots (4), and the relations (31) that the two its components, s_1 and s_2, are the circular splines with these knots.

The computation scheme is the same. If $\mathbf{y}_i$ has already been found ($\mathbf{y}_0$ is given by the initial conditions), we compute $\mathbf{y'}_i$ from (30) and $\mathbf{y}_{i+1}^{(0)}$ by using the explicit Euler formula

$$\mathbf{y}_{i+1}^{(0)} = \mathbf{y}_i + h\,\mathbf{f}(x_i, \mathbf{y}_i)$$

and then, we correct it iteratively using the implicit formula

$$\mathbf{y}_{i+1}^{(k+1)} = \mathbf{y}_i + h\,\mathbf{B}(\mathbf{y}_i', \mathbf{f}(x_{i+1}, \mathbf{y}_{i+1}^{(k)}))$$

for $k = 0, 1, \ldots$ until two successive computations of $\mathbf{y}_{i+1}$ coincide.

In the example we have

$$a = 0, \quad y_{1,0} = 1, \quad y_{2,0} = 0,$$
$$f_1(x, y_1, y_2) = y_2, \quad f_2(x, y_1, y_2) = -\frac{y_2}{x} - y_1.$$

As the result we receive two circular splines, s_1 and s_2, that are approximations of the Bessel J_0 function and its first order derivative J_0', respectively.

Both such circular splines with knots (4) equally spaced in the interval $[0, 10]$ for $n = 20$ and $h = \frac{1}{2}$ are shown in Figure 4. For comparison, the drawing also includes the Bessel J_0 and J_0' functions. As in previous example, the iterative process is equally convergent here, and approximate solutions seem to be accurate despite quite a big step-size.

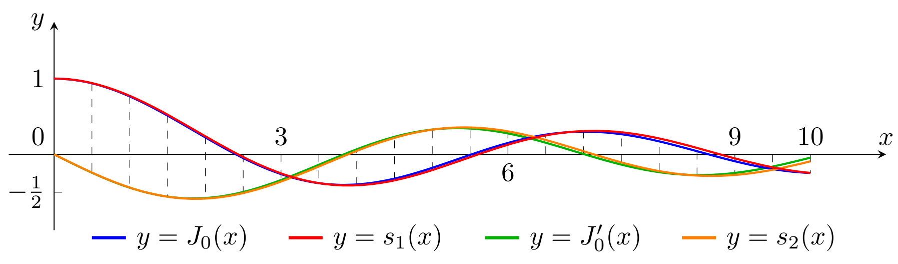

Figure 4: The exact and approximate solutions of (27–28)

Numerical approximations $y_{1,i} = s_1(x_i)$ and $y_{2,i} = s_2(x_i)$ to the values $J_0(x_i)$ and $J_0'(x_i)$ in these knots are given in Table 2. As can be seen, the approximation error is 0.04293 for s_1 and 0.4787 for s_2. Further computations show that each time the step-size is halved, it decreases four-fold.

i	x_i	$y_{1,i}$	$J_0(x_i)$	$J_0(x_i) - y_{1,i}$	$y_{2,i}$	$J_0'(x_i)$	$J_0'(x_i) - y_{2,i}$
0	0.0	1.00000	1.00000	0.00000	0.00000	0.00000	0.00000
1	0.5	0.94082	0.93847	−0.00235	−0.24009	−0.24227	−0.00218
2	1.0	0.77273	0.76520	−0.00753	−0.43820	−0.44005	−0.00185
3	1.5	0.52420	0.51183	−0.01238	−0.55880	−0.55794	0.00086
4	2.0	0.23873	0.22389	−0.01484	−0.58324	−0.57672	0.00651
5	2.5	−0.03453	−0.04838	−0.01385	−0.51090	−0.49709	0.01381
6	3.0	−0.25099	−0.26005	−0.00906	−0.35911	−0.33906	0.02005
7	3.5	−0.37963	−0.38013	−0.00050	−0.16022	−0.13738	0.02284
8	4.0	−0.40806	−0.39715	0.01091	0.04528	0.06604	0.02076
9	4.5	−0.34280	−0.32054	0.02226	0.21767	0.23106	0.01339
10	5.0	−0.20721	−0.17760	0.02961	0.32617	0.32758	0.00141
11	5.5	−0.03705	−0.00684	0.03020	0.35460	0.34144	−0.01317
12	6.0	0.12739	0.15065	0.02325	0.30354	0.27668	−0.02685
13	6.5	0.25027	0.26009	0.00983	0.18947	0.15384	−0.03563
14	7.0	0.30764	0.30008	−0.00756	0.04127	0.00468	−0.03659
15	7.5	0.29145	0.26634	−0.02511	−0.10639	−0.13525	−0.02886
16	8.0	0.20986	0.17165	−0.03821	−0.22101	−0.23464	−0.01362
17	8.5	0.08487	0.04194	−0.04293	−0.27933	−0.27312	0.00621
18	9.0	−0.05286	−0.09033	−0.03747	−0.27161	−0.24531	0.02630
19	9.5	−0.17135	−0.19393	−0.02258	−0.20288	−0.16126	0.04161
20	10.0	−0.24468	−0.24594	−0.00125	−0.09135	−0.04347	0.04787

Table 2: Approximate solutions of (27-28) and their accuracy

Let us now examine the Lipschitz condition for the differential equation in question. Due to the fact that it is a two-dimensional case, instead of the number module we use the vector norm

$$\|\mathbf{v}\| = \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\| = \max\left(|v_1|, |v_2|\right).$$

Applying two-dimensional version of mean-value theorem, for any $x > 0$, $\mathbf{y}$, $\mathbf{y}^*$ and some ζ_1, η_1, ζ_2, η_2 we get

$$\begin{aligned}
\|\mathbf{f}(x,\mathbf{y}) \; - \; \mathbf{f}(x,\mathbf{y}^*)\| &= \\
&= \left\| \begin{bmatrix} f_1(x,y_1,y_2) \\ f_2(x,y_1,y_2) \end{bmatrix} - \begin{bmatrix} f_1(x,y_1^*,y_2^*) \\ f_2(x,y_1^*,y_2^*) \end{bmatrix} \right\| = \\
&= \left\| \begin{bmatrix} f_1(x,y_1,y_2) - f_1(x,y_1^*,y_2^*) \\ f_2(x,y_1,y_2) - f_2(x,y_1^*,y_2^*) \end{bmatrix} \right\| = \\
&= \left\| \begin{bmatrix} \dfrac{\partial f_1(x,\zeta_1,\eta_1)}{\partial y_1}(y_1 - y_1^*) + \dfrac{\partial f_1(x,\zeta_1,\eta_1)}{\partial y_2}(y_2 - y_2^*) \\ \dfrac{\partial f_2(x,\zeta_2,\eta_2)}{\partial y_1}(y_1 - y_1^*) + \dfrac{\partial f_2(x,\zeta_2,\eta_2)}{\partial y_2}(y_2 - y_2^*) \end{bmatrix} \right\| = \\
&= \left\| \begin{bmatrix} 0 \cdot (y_1 - y_1^*) \; + \; 1 \cdot (y_2 - y_2^*) \\ -1 \cdot (y_1 - y_1^*) \; - \; \dfrac{1}{x} \cdot (y_2 - y_2^*) \end{bmatrix} \right\| = \max\left(1, \frac{1}{x}\right) \|\mathbf{y} - \mathbf{y}^*\|.
\end{aligned}$$

The result shows that in view of Theorem 2 the step-size $h = \frac{1}{2}$ is on the verge of convergence of the method for $x \geq 1$, and is inappropriate for $x < 1$.

Error estimates

It is well known that under certain regularity conditions of the function f there exists a unique continuously differentiable function $y = y(x)$, defined on the closed interval $[a, b]$, that satisfies the initial value problem (1–2). In turn, Theorem 2 guarantees that for a sufficiently small step-size h there exists exactly one approximate solution of (1–2) in the form of a circular spline $y = s(x)$ with knots (4) equally spaced in $[a, b]$. This approximation satisfies the equalities (9) and (22), so it can be found iteratively using the implicit formula

$$y_{i+1} = y_i + hB(f(x_i, y_i), f(x_{i+1}, y_{i+1})) \quad (i = 0, 1, \ldots, n-1).$$

To assess the quality of the method we compare these two functions in the knots (4) and in the entire interval $[a, b]$. First we estimate the error of the approximate solution at the knots, comparing the values $y(x_i)$ of the exact solution with the numerical approximations $y_i = s(x_i)$:

$$e_i = y(x_i) - y_i \quad (i = 0, 1, \ldots, n). \tag{32}$$

Theorem 3. *If $f \in C^2$ and there exist constatnts M, N_x and N_y such that*

$$|f(x,y)| \le M, \qquad \left|\frac{\partial f(x,y)}{\partial x}\right| \le N_x, \qquad \left|\frac{\partial f(x,y)}{\partial y}\right| \le N_y$$

and a constant Y_3 such that for the third derivative of the exact solution the inequality

$$|y'''(x)| \le Y_3$$

holds for $x \in [a, b]$, then for step-size $h < \frac{1}{2L}$

$$|y(x_i) - y_i| \le Kh^2, \quad |y'(x_i) - y'_i| \le LKh^2 \quad (i = 0, 1, \ldots, n), \tag{33}$$

where L is a constant in the Lipschitz condition (3) *and*

$$K = \frac{Y_3 + 9[N_x + N_y(1 + 2M)]^2}{12L}\left[e^{\frac{4}{3}L(b-a)} - 1\right].$$

Proof. From (10) and (19) we have for any u

$$B(u, u) = u, \qquad \frac{\partial B(u, u)}{\partial v} = \frac{1}{2}.$$

So, letting $i \in \{0, 1, \ldots, n-1\}$, by applying Taylor's theorem, it follows from (32) that for a certain η_i

$$\begin{aligned}
e_{i+1} &= y(x_{i+1}) - y_{i+1} = y(x_{i+1}) - y_i - hB(y'_i, y'_{i+1}) = \\
&= y(x_i) - y_i + y(x_{i+1}) - y(x_i) - \\
&\quad -h\left[B(y'_i, y'_i) + \frac{\partial B(y'_i, y'_i)}{\partial v}(y'_{i+1} - y'_i) + \frac{\partial^2 B(y'_i, s'(\eta_i))}{\partial v^2}\frac{(y'_{i+1} - y'_i)^2}{2}\right] = \\
&= e_i + y(x_{i+1}) - y(x_i) - \frac{h}{2}(y'_{i+1} + y'_i) - h\frac{\partial^2 B(y'_i, s'(\eta_i))}{\partial v^2}\frac{(y'_{i+1} - y'_i)^2}{2} = \\
&= e_i + \frac{h}{2}\left[y'(x_{i+1}) - y'_{i+1} + y'(x_i) - y'_i\right] + y(x_{i+1}) - y(x_i) - \\
&\quad -\frac{h}{2}\left[y'(x_{i+1}) + y'(x_i)\right] - h\frac{\partial^2 B(y'_i, s'(\eta_i))}{\partial v^2}\frac{(y'_{i+1} - y'_i)^2}{2}.
\end{aligned}$$

We now investigate consecutive components of the above expression. And so, in view of Lipshitz condition (3), we get

$$|y'(x_{i+1}) - y'_{i+1}| \le L|e_{i+1}|, \quad |y'(x_i) - y'_i| \le L|e_i|. \tag{34}$$

Next, we have

$$\left| y(x_{i+1}) - y(x_i) - \frac{h}{2}\left[y'(x_{i+1}) + y'(x_i)\right] \right| \le \frac{1}{12}h^2 Y_3, \tag{35}$$

as the expression within the modulus symbols represents exactly the error which is produced by trapezoidal rule integrating the function y' within the limits x_i and x_{i+1}, and which equals to $-\frac{1}{12}h^3 y'''(\xi_i)$ (see [3], p. 459). At last, applying two-variable version of mean-value theorem, for some γ_i and ζ_i we obtain

$$\begin{aligned} y'_{i+1} - y'_i &= f(x_{i+1}) - f(x_i) = \\ &= \frac{\partial f(\gamma_i, s(\zeta_i))}{\partial u} h + \frac{\partial f(\gamma_i, s(\zeta_i))}{\partial v}(y_{i+1} - y_i) = \\ &= h\left[\frac{\partial f(\gamma_i, s(\zeta_i))}{\partial u} + \frac{\partial f(\gamma_i, s(\zeta_i))}{\partial v} B(y'_i, y'_{i+1})\right], \end{aligned}$$

which, in relation to the inequality

$$|B(y'_i, y'_{i+1})| \le |y'_i| + \sqrt{1 + {y'_i}^2} \le 1 + 2|y'_i| = 1 + 2|f(x_i, y_i)|,$$

resulting from (16), leads to

$$|y'_{i+1} - y'_i| \le h\left[N_x + N_y(1 + 2M)\right]. \tag{36}$$

Continuing to estimate the error of the method, after introducing

$$T = \frac{1}{12}Y_3 + \frac{3}{4}\left[N_x + N_y(1 + 2M)\right]^2,$$

and taking into account (34), (35), (36) and (18), we get

$$|e_{i+1}| \le |e_i| + \frac{Lh}{2}(|e_{i+1}| + |e_i|) + Th^3,$$

and hence

$$|e_{i+1}|\left(1 - \frac{Lh}{2}\right) \le |e_i|\left(1 + \frac{Lh}{2}\right) + Th^3.$$

Since the assumption $0 < h < \frac{1}{2L}$ it follows that

$$\frac{3}{4} < 1 - \frac{Lh}{2}. \tag{37}$$

This implies that $1 - Lh/2 > 0$, so we receive the recursive dependence

$$|e_{i+1}| \le |e_i|\frac{1 + Lh/2}{1 - Lh/2} + \frac{T}{1 - Lh/2}h^3.$$

Using this inequality we can easily express the estimate of the error by known quantities. Indeed, introducing

$$C = \frac{1 + Lh/2}{1 - Lh/2}, \quad D = \frac{T}{1 - Lh/2},$$

we have

$$|e_{i+1} \leq C|e_i| + Dh^3.$$

Then, since $e_0 = 0$, this implies that

$$\begin{aligned}
|e_1| &\leq Dh^3, \\
|e_2| &\leq (1+C)Dh^3, \\
|e_3| &\leq (1+C+C^2)Dh^3, \\
\cdots & \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
|e_i| &\leq (1+C+C^2+\ldots+C^{i-1})Dh^3 = \frac{C^i-1}{C-1}Dh^3.
\end{aligned}$$

Now, letting $i \in \{0,1,\ldots,n\}$ and replacing C and D by the previously assigned expressions, we get

$$|e_i| \leq \frac{\left(\dfrac{1+Lh/2}{1-Lh/2}\right)^i - 1}{\dfrac{1+Lh/2}{1-Lh/2} - 1}\frac{Th^3}{1-Lh/2} = \left[\left(1+\frac{Lh}{1-Lh/2}\right)^i - 1\right]\frac{T}{L}h^2.$$

Thus, in view of (37) and the fact that $1+u < \mathrm{e}^u$ for any $u > 0$, we deduce that

$$\begin{aligned}
|e_i| &\leq \left[\left(1+\frac{4}{3}Lh\right)^i - 1\right]\frac{T}{L}h^2 \leq \left[\left(1+\frac{\frac{4}{3}L(x_i-x_0)}{i}\right)^i - 1\right]\frac{T}{L}h^2 \leq \\
&\leq \left[\mathrm{e}^{\frac{4}{3}L(x_i-x_0)} - 1\right]\frac{T}{L}h^2 \leq Kh^2.
\end{aligned}$$

This gives the first estimate of (33). The second of (33) is a consequence the Lipschitz condition (3), namely

$$|y'(x_i) - y_i'| = |f(x_i, y(x_i)) - f(x_i, y_i)| \leq L|y(x_i) - y_i| \leq LKh^2.$$

That completes the proof. □

Our intention is now to estimate the error of the approximate solution as well as the errors of its first and second derivatives over the entire interval $[a,b]$. But the function s'' may be unspecified in internal knots (4) of $[a,b]$. If so, we define it there as

$$s''(x_i) = \frac{1}{2}\left[\lim_{x\to x_i-} s''(x) + \lim_{x\to x_i+} s''(x)\right] \quad (i=1,2,\ldots,n-1). \tag{38}$$

Theorem 4. *If the assumption of Theorem* 3 *are satisfied, then for step-size* $h < \frac{1}{2L}$ *the inequalities*

$$|y(x) - s(x)| \leq K_0h^2, \quad |y'(x) - s'(x)| \leq K_1h^2,$$

$$|y''(x) - s''(x)| \leq K_2h$$

hold for all $x \in [a,b]$, *where*

$$K_2 = Y_3 + 2LK + 3M(1+M^2)[N_x + N_y(1+2M)]^2,$$

$$K_0 = \frac{6K + K_2/L}{4}, \quad K_1 = LK + K_2.$$

Proof. Let $i \in \{0, 1, \dots, n-1\}$. Since the function s is a circular spline with knots (4), it is given in $[x_i, x_{i+1}]$ by the circular arc lying either on the upper ($z = -1$) or lower ($z = 1$) half of a circle. Let r be the radius of this circle. Then, in view of (12) and (20), we obtain

$$\begin{aligned}
\frac{1}{r} &= \frac{1}{h}\left|\frac{y'_{i+1}}{\sqrt{1+y'^2_{i+1}}} - \frac{y'_i}{\sqrt{1+y'^2_i}}\right| = \\
&= \frac{\left|y'_{i+1}\sqrt{1+y'^2_i} - y'_i\sqrt{1+y'^2_{i+1}}\right|}{h\sqrt{1+y'^2_i}\sqrt{1+y'^2_{i+1}}} = \\
&= \frac{\left|y'_{i+1}\sqrt{1+y'^2_i} - y'_i\sqrt{1+y'^2_{i+1}}\right|\left(\sqrt{1+y'^2_i}+\sqrt{1+y'^2_{i+1}}\right)}{h\sqrt{1+y'^2_i}\sqrt{1+y'^2_{i+1}}\left(\sqrt{1+y'^2_i}+\sqrt{1+y'^2_{i+1}}\right)} \leq \\
&\leq \frac{\left|y'_{i+1}\left(1+y'^2_i\right) + \left(y'_{i+1}-y'_i\right)\sqrt{1+y'^2_i}\sqrt{1+y'^2_{i+1}} - y'_i\left(1+y'^2_{i+1}\right)\right|}{2h\sqrt{1+y'^2_i}\sqrt{1+y'^2_{i+1}}} \leq \\
&\leq \frac{\left|y'_{i+1}-y'_i\right|\left(1+\left|y'_i y'_{i+1}\right| + \sqrt{1+y'^2_i}\sqrt{1+y'^2_{i+1}}\right)}{2h\sqrt{1+y'^2_i}\sqrt{1+y'^2_{i+1}}} \leq \frac{\left|y'_{i+1}-y'_i\right|}{h}.
\end{aligned}$$

Thus, if $x, x^* \in (x_i, x_{i+1})$, then by the formula defining the curvature of the function

$$s''(x) = \frac{1}{zr}\sqrt{1+[s'(x)]^2}^{\,3}$$

and the equalities

$$|s'(x)| \leq \max\left\{|y'_i|, |y'_{i+1}|\right\} \leq M, \quad |s'(x) - s'(x^*)| \leq |y'_{i+1} - y'_i|,$$

arising from the convexity of the circle arc in question, we can write

$$\begin{aligned}
|s''(x) - s''(x^*)| &= \frac{1}{r}\left|\sqrt{1+[s'(x)]^2}^{\,3} - \sqrt{1+[s'(x^*)]^2}^{\,3}\right| \leq \\
&\leq \frac{|y'_{i+1}-y'_i|}{h}\left|\sqrt{1+[s'(x)]^2} - \sqrt{1+[s'(x^*)]^2}\right|\Big\{1+[s'(x)]^2 + \\
&\quad + \sqrt{1+[s'(x)]^2}\sqrt{1+[s'(x^*)]^2} + 1 + [s'(x^*)]^2\Big\} \leq \\
&\leq 3(1+M^2)\frac{|y'_{i+1}-y'_i|}{h}\frac{\left|[s'(x)]^2 - [s'(x^*)]^2\right|}{\sqrt{1+[s'(x)]^2}+\sqrt{1+[s'(x^*)]^2}} \leq \\
&\leq 3M(1+M^2)\frac{|y'_{i+1}-y'_i|}{h}.
\end{aligned}$$

Hence, by (36)

$$|s''(x) - s''(x^*)| \leq 3M(1+M^2)[N_x + N_y(1+2M)]h. \tag{39}$$

Further, by mean-value theorem, for some $\xi_{1,i}, \xi_{2,i} \in (x_i, x_{i+1})$ we have

$$y'(x_{i+1}) = y'(x_i) + hy''(\xi_{1,i}), \quad y'_{i+1} = y'_i + hs''(\xi_{1,i}).$$

Combining these and the second of (33), we get

$$|y''(\xi_{1,i}) - s''(\xi_{2,i})| \le \frac{1}{h}\left[\,|y'(x_i) - y'_i| + |y'(x_{i+1}) - y'_{i+1}|\,\right] \le 2LKh.$$

So, this and (39) imply that

$$\begin{aligned} |y''(x) \; - \; s''(x)| &\le \\ &\le |y''(x) - y''(\xi_{1,i})| + |y''(\xi_{1,i}) - s''(\xi_{2,i})| + |s''(\xi_{2,i}) - s''(x)| \le \\ &\le \left\{ Y_3 + 2LK + 3M(1+M^2)[N_x + N_y(1+2M)]^2 \right\} h = K_2 h \qquad (40) \end{aligned}$$

fullfils for any $x \in (x_i, x_{i+1})$. By (38) the above inequality holds also in interior knots (4), which establishes the third part of the thesis.

At the end, by Taylor's theorem, for any $x \in [x_i, x_{i+1}]$ there exist $\eta_{1,i}, \eta_{2,i} \in (x_i, x_{i+1})$ such that

$$|y(x) - s(x)| \le |y(x_i) - y_i| + h|y'(x_i) - y'_i| + \frac{h^2}{2!}|y''(\eta_{1,i}) - s''(\eta_{1,i})|$$

and

$$|y'(x) - s'(x)| \le |y'(x_i) - y'_i| + h|y''(\eta_{2,i}) - s''(\eta_{2,i})|.$$

Therefore, by (33) and (40), and the assumption $0 < h < \frac{1}{2L}$, we obtain

$$|y(x) - s(x)| \le Kh^2 + LKh^3 + \frac{1}{2}K_2h^3 \le \frac{1}{4}\left(6K + \frac{K_2}{L}\right)h^2 = K_0h^2$$

and

$$|y'(x) - s'(x)| \le LKh^2 + K_2h^2 = (LK + K_2)h^2 = K_1h^2$$

which are the first and second estimation in the thesis.

It should be noted that the above considerations were carried out with the default assumption $r < \infty$. A moment of reflection is enough to make sure that in the case of $r = \infty$ they lead, with some simplifications, to the same result. □

Rounding and iteration stopping errors

Let $\tilde{y}_i$ be approximate values, which we get in reality instead of y_i $(i = 0, 1, \ldots, n)$. We have then

$$\tilde{y}_{i+1} = \tilde{y}_i + hB(f(x_i, \tilde{y}_i), f(x_{i+1}, y^*_{i+1})) + \tilde{e}_{i+1} \quad (i = 0, 1, \ldots, n-1), \qquad (41)$$

where y^*_{i+1} is the solution to the equation

$$y^*_{i+1} = \tilde{y}_i + hB(f(x_i, \tilde{y}_i), f(x_{i+1}, y^*_{i+1})), \qquad (42)$$

which we solve iteratively instead of (24), and $\tilde{e}_{i+1}$ is a local error of interrupting iterations and rounding. The error caused by interrupting the iterations can be made arbitrarily small, assuming a sufficiently large number of iterations.

We are interested in estimating the values

$$d_i = y_i - \tilde{y}_i \quad (i = 0, 1, \dots, n).$$

Letting $i \in \{0, 1, \dots, n-1\}$, in view of (3), (17), (24), (41) and (42) we have

$$\begin{aligned}
|d_{i+1}| &= |y_{i+1} - y^*_{i+1} - \tilde{e}_{i+1}| = \\
&= |y_i + hB(f(x_i, y_i), f(x_{i+1}, y_{i+1})) - \tilde{y}_i - hB(f(x_i, \tilde{y}_i), f(x_{i+1}, y^*_{i+1})) - \\
&\quad -\tilde{e}_{i+1}| \leq \\
&\leq |d_i| + 2h \left[|f(x_i, y_i) - f(x_i, \tilde{y}_i)| + |f(x_{i+1}, y_{i+1}) - f(x_{i+1}, y^*_{i+1})|\right] + \\
&\quad +|\tilde{e}_{i+1}| \leq \\
&\leq |d_i| + 2Lh\,(|d_i| + |d_{i+1}| + |\tilde{e}_{i+1}|) + |\tilde{e}_{i+1}|.
\end{aligned}$$

Therefore

$$|d_{i+1}| \leq \frac{1 + 2Lh}{1 - 2Lh}\,(|d_i| + |\tilde{e}_{i+1}|)\,.$$

Assuming that for some $E > 0$

$$|\tilde{e}_i| \leq E \quad (i = 1, 2, \dots, n),$$

and introducing

$$C = \frac{1 + Lh/2}{1 - Lh/2},$$

we get

$$|d_{i+1}| \leq C\,(|d_i| + E) \quad (i = 0, 1, \dots, n-1).$$

Hence, in view of equality $d = 0$, we deduce that

$$\begin{aligned}
|d_1| &\leq CE, \\
|d_2| &\leq (1 + C)CE, \\
|d_3| &\leq (1 + C + C^2)CE, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
|d_i| &\leq (1 + C + C^2 + \dots + C^{i-1})CE = \frac{C^i - 1}{C - 1}CE.
\end{aligned}$$

Replacing C by the previously assigned expression, we obtain

$$|d_i| \leq \frac{\left(\dfrac{1 + 2Lh}{1 - 2Lh}\right)^i - 1}{\dfrac{1 + 2Lh}{1 - 2Lh} - 1}\frac{1 + 2Lh}{1 - 2Lh}E = \left[\left(1 + \frac{4Lh}{1 - 2Lh}\right)^i - 1\right]\frac{1 + 2Lh}{4Lh}E.$$

Since $h < \frac{1}{2L}$ this implies that $2Lh < 1$, so by virtue of inequality $1 + u < \mathrm{e}^u$ satisfied for any $u > 0$, we finally get

$$\begin{aligned}
|d_i| &\leq \left[\left(1 + \frac{4L(x_i - x_0)}{i(1 - 2Lh)}\right)^i - 1\right]\frac{E}{2Lh} \leq \left[\mathrm{e}^{\frac{4L(x_i - x_0)}{1 - 2Lh}} - 1\right]\frac{E}{2Lh} \leq \\
&\leq \left[\mathrm{e}^{\frac{4L(b - a)}{1 - 2Lh}} - 1\right]\frac{E}{2Lh} \qquad\qquad (i = 0, 1, \dots, n).
\end{aligned}$$

This estimation is pessimistic for very small h, as well as for h very close to $\frac{1}{2L}$. The first case seems obvious. After all, the number of roundings and the number of iteration interruptions increases to infinity when the step-size tends to zero, so a total error may grow dramatically. The error behavior in the second case is not so intuitively perceptible. Perhaps this phenomenon is related to the convergence of the method.

Conclusions

The presented method has a theoretical meaning, as it is an example of a method based on a non-linear differentiation formula. Its practical utility is limited due to the low order and higher computational complexity than its equivalent linear methods. However, in special cases, it may be applicable, for example in numerical control of drawing and machining devices, in which the tool moves along a circular arc.

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